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Existence of countably many positive solutions of nth-order m-point boundary value problems[☆]

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ABSTRACT

The fixed point index theory and a new fixed point theorem in cones are used to prove the existence of countably many positive solutions of nth-order m-point nonlinear boundary value problems.

1. Introduction

In this paper, we introduce a new operator, which improves and generates a p-Laplace operator for some $p > 1$, and we study the existence of countably many positive solutions of nth-order m-point nonlinear boundary value problems of the form

$$(\varphi(p(t)u^{(n-1)})(t))' + a(t)f(u(t), u'(t), \ldots, u^{(n-2)}(t)) = 0, \quad 0 < t < 1, \tag{1.1}$$

with the boundary value conditions

$$\begin{cases} u^{(i)}(0) = 0, \quad i = 0, 1, \ldots, n-3, \\ u^{(n-2)}(0) = \sum_{i=1}^{m-2} \alpha_i u^{(n-2)}(\xi_i), \qquad u^{(n-1)}(1) = 0, \end{cases} \tag{1.2}$$

where $\varphi : R \to R$ is an increasing homeomorphism and positive homomorphism and $\varphi(0) = 0$. Here $\xi_i \in (0, 1)$ with $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1$ and α_i satisfies $\alpha_i \in [0, +\infty)$, $0 < \sum_{i=1}^{m-2} \alpha_i < 1$, $p \in C([0, 1], (0, +\infty))$, $f \in C([0, +\infty)^{n-1}, [0, +\infty))$, $a : [0, 1] \to [0, +\infty)$ and has countably many singularities in $[0, \frac{1}{2})$.

A projection $\varphi : R \to R$ is called an increasing homeomorphism and positive homomorphism if the following conditions are satisfied:

(1) if $x \le y$, then $\varphi(x) \le \varphi(y)$, for all $x, y \in R$;
(2) φ is a continuous bijection and its inverse is also continuous;

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(3) $\varphi(xy) = \varphi(x)\varphi(y)$, for all $x, y \in R_+$.

In the above definition, condition (3) can be replaced by the following stronger condition:

(4) $\varphi(xy) = \varphi(x)\varphi(y)$, for all $x, y \in R$, where $R = (-\infty, +\infty)$.

Remark 1.1. If conditions (1), (2) and (4) hold, then φ is homogenous and generates a p-Laplace operator, i.e., $\varphi(x) = |x|^{p-2}x$, for some $p > 1$.

Remark 1.2. It is well known that a p-Laplacian operator is odd. However, the operator which we defined above is not necessary odd, see Example 5.2.

Recently, the existence and multiplicity of positive solutions for the second order and p-Laplacian operator with multi-point boundary value problems, i.e., $p(t) \equiv 1$ and $\varphi(x) = x$, $\varphi(x) = |x|^{p-2}x$, for some $p > 1$, have received wide attention, see [1–3,5–7,9,10] and references therein. We know that the oddness of a p-Laplacian operator is key to the proof. However, in this paper we define a new operator which improves and generates a p-Laplacian operator for some $p > 1$ and φ that is not necessarily odd. Moreover, for increasing homeomorphism and positive homomorphism, operator and research has proceeded very slowly, see [8,12]. In particular, the existence of countably many positive solutions for nth-order m-point boundary value problems still remains unknown.

In [11], Liu and Zhang studied the existence of positive solutions of quasi-linear differential equation of the form

$$\begin{aligned}&(\varphi(x'))' + a(t)f(x(t)) = 0, \quad 0 < t < 1,\\ &x(0) - \beta x'(0) = 0, \qquad x(1) + \delta x'(1) = 0,\end{aligned}$$

where $\varphi : R \to R$ is an increasing homeomorphism and positive homomorphism and $\varphi(0) = 0$. They proved the existence of one or two positive solutions by using a fixed point index theorem in cones.

In [3], Zhou and Su studied the quasi-linear equation with a p-Laplacian operator

$$\begin{cases}(\phi_p(u^{(n-1)}))' + g(t)f(u(t), u'(t), \ldots, u^{(n-2)}(t)) = 0, \quad 0 < t < 1,\\ u^{(i)}(0) = 0, \quad 0 \le i \le n-3,\\ u^{(n-1)}(0) - B_0(u^{(n-1)}(\xi)) = 0, \quad n \ge 3,\\ u^{(n-1)}(0) - B_1(u^{(n-1)}(\eta)) = 0, \quad n \ge 3,\end{cases}$$

where $\phi(s)$ is a p-Laplacian operator. They used the fixed point index theory to find conditions for the existence of one solution, and of multiple solutions. We emphasize that the results of the paper [3] are not replaced by φ which we defined above.

But whether or not we can obtain countably many positive solutions of the nth-order m-point boundary value problem (1.1) and (1.2) still remains unknown. So the goal of the present paper is to improve and generate a p-Laplacian operator and establish some criteria for the existence of countably many solutions.

We shall assume that the following conditions are satisfied.

(C_1) $\varphi : R \to R$ is an increasing homeomorphism and positive homomorphism;
(C_2) $f \in C([0, +\infty)^{n-1}, [0, +\infty))$, and α_i satisfies $0 < \sum_{i=1}^{m-2} \alpha_i < 1$;
(C_3) $p \in C([0, 1], (0, +\infty))$ is a nondecreasing function;
(C_4) There exists a sequence $\{t_i\}_{i=1}^{\infty}$ such that $t_{i+1} < t_i$, $t_1 < \frac{1}{2}$, $\lim_{i\to\infty} t_i = t_0 > 0$, and $\lim_{t\to t_i} a(t) = \infty$, $i = 1, 2, \ldots$, and

$$0 < \int_0^1 a(t)\mathrm{d}t < +\infty.$$

Moreover, $a(t)$ does not vanish identically on any subinterval of $[0, 1]$.

The plan of the paper is as follows. In Section 2, for the convenience of the reader we give some definitions. In Section 3, we present some lemmas in order to prove our main results. Section 4 is developed to present and prove our main results. In Section 5 we present two examples of increasing homeomorphism and positive homomorphism operators.

2. Some definitions and fixed point theorems

In this section, we provide background definitions from the cone theory in Banach spaces.

Definition 2.1. Let $(E, \|.\|)$ be a real Banach space. A nonempty, closed, convex set $P \subset E$ is said to be a cone provided the following are satisfied:

(a) if $y \in P$ and $\lambda \ge 0$, then $\lambda y \in P$;
(b) if $y \in P$ and $-y \in P$, then $y = 0$.

If $P \subset E$ is a cone, we denote the order induced by P on E by $\le$, that is, $x \le y$ if and only if $y - x \in P$.

Definition 2.2. A map α is said to be a nonnegative, continuous, concave functional on a cone P of a real Banach space E, if

$$\alpha : P \to [0, \infty)$$

is continuous, and

$$\alpha(tx + (1-t)y) \geq t\alpha(x) + (1-t)\alpha(y)$$

for all $x, y \in P$ and $t \in [0, 1]$.

Definition 2.3. Given a nonnegative continuous functional γ on a cone P of E, for each $d > 0$ we define the set

$$P(\gamma, d) = \{x \in P : \gamma(x) < d\}.$$

The following fixed point theorems are fundamental and important for the proofs of our main results.

Theorem 2.1 ([4]). *Let E be a Banach space and $P \subset E$ be a cone in E. Let $r > 0$ define $\Omega_r = \{x \in P : \|x\| < r\}$. Assume that $T : P \bigcap \overline{\Omega}_r \to P$ is a completely continuous operator such that $Tx \neq x$ for $x \in \partial\Omega_r$.*

(i) *If $\|Tx\| < \|x\|$ for $x \in \partial\Omega_r$, then*

$$i(T, \Omega_r, P) = 1.$$

(ii) *If $\|Tx\| > \|x\|$ for $x \in \partial\Omega_r$, then*

$$i(T, \Omega_r, P) = 0.$$

Theorem 2.2 ([9]). *Let P be a cone in a Banach space E. Let α, β and γ be three increasing, nonnegative and continuous functionals on P, satisfying*

$$\gamma(x) \leq \beta(x) \leq \alpha(x), \qquad \|x\| \leq M\gamma(x),$$

for some $c > 0$ and $M > 0$ and all $x \in \overline{P(\gamma, c)}$. Suppose there exists a completely continuous operator $T : \overline{P(\gamma, c)} \to P$ and $0 < a < b < c$ such that

(i) *$\gamma(Tx) < c$, for all $x \in \partial P(\gamma, c)$;*
(ii) *$\beta(Tx) > b$, for all $x \in \partial P(\beta, b)$;*
(iii) *$P(\alpha, a) \neq \emptyset$, and $\alpha(Tx) < a$, for all $x \in \partial P(\alpha, a)$.*

Then T has at least three fixed points $x_1, x_2, x_3 \in \overline{P(\gamma, c)}$ such that

$$0 \leq \alpha(x_1) < a < \alpha(x_2), \qquad \beta(x_2) < b < \beta(x_3), \qquad \gamma(x_3) < c.$$

3. Preliminaries and lemmas

Let $E = \{u \in C^{n-2}[0, 1] : u^{(i)}(0) = 0, i = 0, 1, \ldots, n-3\}$. Thus E is a Banach space when endowed with the norm $\|u\| = \sup_{t\in[0,1]} |u^{(n-2)}(t)|$. Now let

$$K = \left\{u \in E : u^{(n-2)}(t) \text{ is a nondecreasing and nonnegative concave function on } [0, 1]\right\}.$$

Obviously K is a cone in E.

We can easily get the following lemmas.

Lemma 3.1. *Suppose* (C_4) *holds. Then we have the following conclusions:*

(i) *for each constant $\theta \in (0, \frac{1}{2})$, we have*

$$0 < \int_\theta^{1-\theta} a(t)\mathrm{d}t < +\infty,$$

(ii) *the function*

$$H(t) = \int_t^{1-t_1} \frac{1}{p(s)}\varphi^{-1}\left(\int_s^{1-t_1} a(\tau)\mathrm{d}\tau\right)\mathrm{d}s + \frac{\sum_{i=1}^{m-2}\alpha_i \int_{t_1}^{t} \frac{1}{p(s)}\varphi^{-1}\left(\int_s^t a(\tau)\mathrm{d}\tau\right)\mathrm{d}s}{1 - \sum_{i=1}^{m-2}\alpha_i}$$

is continuous and positive on $[t_1, 1-t_1]$. Furthermore,

$$L = \min_{t\in[t_1, 1-t_1]} H(t) > 0.$$

Proof. Firstly, we can easily obtain (i) from condition (C_4).

Next, we prove conclusion (ii). It is easily seen that $H(t)$ is continuous on $[t_1, 1-t_1]$. Let

$$H_1(t) = \int_t^{1-t_1} \frac{1}{p(s)}\varphi^{-1}\left(\int_s^{1-t_1} a(\tau)\mathrm{d}\tau\right)\mathrm{d}s,$$

$$H_2(t) = \frac{\sum_{i=1}^{m-2}\alpha_i \int_{t_1}^{t} \frac{1}{p(s)}\varphi^{-1}\left(\int_s^t a(\tau)\mathrm{d}\tau\right)\mathrm{d}s}{1-\sum_{i=1}^{m-2}\alpha_i}.$$

Then from condition (C_4), we know that $H_1(t)$ is strictly monotone decreasing on $[t_1, 1-t_1]$ and $H_1(1-t_1) = 0$. Similarly, the function $H_2(t)$ is strictly monotone increasing on $[t_1, 1-t_1]$ and $H_2(t_1) = 0$. So the function $H(t) = H_1(t) + H_2(t)$ is positive on $[t_1, 1-t_1]$, which implies $L = \min_{t\in[t_1,1-t_1]} H(t) > 0$. □

Lemma 3.2 ([7]). *Let $u \in K$ and θ of Lemma 3.1, then*

$$u^{(n-2)}(t) \geq \theta\|u\|, \quad t \in [\theta, 1-\theta].$$

Lemma 3.3 ([3]). *Suppose that conditions (C_2)–(C_4) hold. Then the solution $u(t)$ of problem (1.1), (1.2) satisfies:*

$$u(t) \leq u'(t) \leq \cdots \leq u^{(n-1)}(t), \quad 0 \leq t \leq 1,$$

and for any $\theta \in (0, \frac{1}{2})$ in Lemma 3.2, we have

$$u^{(n-3)}(t) \leq \frac{1}{\theta}u^{(n-2)}(t), \quad \theta \leq t \leq 1-\theta.$$

Now, we define an operator $T : K \to E$ by

$$(Tu)(t) = \int_0^t \int_0^{\zeta_1} \cdots \int_0^{\zeta_{n-3}} w(\zeta_{n-2})\mathrm{d}\zeta_{n-2}\mathrm{d}\zeta_{n-3}\cdots\mathrm{d}\zeta_1, \tag{3.1}$$

where

$$w(t) = \int_0^t \frac{1}{p(s)}\varphi^{-1}\left(\int_s^1 a(\tau)f(u(\tau), u'(\tau), \ldots, u^{(n-2)}(\tau))\mathrm{d}\tau\right)\mathrm{d}s + \frac{\sum_{i=1}^{m-2}\alpha_i \int_0^{\xi_i} \frac{1}{p(s)}\varphi^{-1}\left(\int_s^1 a(\tau)f(u(\tau), u'(\tau), \ldots, u^{(n-2)}(\tau))\mathrm{d}\tau\right)\mathrm{d}s}{1-\sum_{i=1}^{m-2}\alpha_i}.$$

Then it is easy to see that $(Tu)^{(n-2)}(t) \geq 0$ $(0 \leq t \leq 1)$, $\varphi[p(t)(Tu)^{(n-1)}](t) \geq 0$ and $[\varphi(p(t)(Tu)^{(n-1)}(t))]' = -a(t)f(u(t), u'(t), \ldots, u^{(n-2)}(t)) \leq 0$, together with $p(t)$ a nondecreasing function, we know $(Tu)^{(n-2)}$ is a concave function. This shows that $T(K) \subset K$.

The following lemma follows from the Arzela–Ascoli Theorem.

Lemma 3.4. *Let conditions (C_1)–(C_4) hold. Then $T : K \to K$ is completely continuous.*

4. Main results

For notational convenience, we let

$$\lambda_1 = \frac{1}{L}, \qquad \lambda_2 = \frac{p(0)(1-\sum_{i=1}^{m-2}\alpha_i)}{\int_0^1 \varphi^{-1}\left(\int_s^1 a(\tau)\mathrm{d}\tau\right)\mathrm{d}s}.$$

The main results of this paper are the following.

Theorem 4.1. *Suppose that conditions (C_1)–(C_4) hold. Let $\{\theta_k\}_{k=1}^{\infty}$ be such that $\theta_k \in (t_{k+1}, t_k)$ $(k = 1, 2, \ldots)$. Let $\{r_k\}_{k=1}^{\infty}$ and $\{R_k\}_{k=1}^{\infty}$ be such that*

$$R_{k+1} < \theta_k r_k < r_k < mr_k < R_k, \quad k = 1, 2, \ldots.$$

Furthermore, for each natural number k we assume that f satisfies:

(C_5) $f(v_1, v_2, \ldots, v_{n-1}) \geq \varphi(mr_k)$, *for all* $0 \leq v_1, v_2, \ldots, v_{n-2} \leq r_k/\theta_k$, $\theta_k r_k \leq v_{n-1} \leq r_k$,
(C_6) $f(v_1, v_2, \ldots, v_{n-1}) \leq \varphi(MR_k)$, *for all* $0 \leq v_1, v_2, \ldots, v_{n-1} \leq R_k$,

where $m \in (\lambda_1, \infty)$ and $M \in (0, \lambda_2)$. Then the boundary value problem (1.1) and (1.2) has infinitely many solutions $\{u_k\}_{k=1}^{\infty}$ such that

$$r_k \leq \|u_k\| \leq R_k, \quad k = 1, 2, \ldots .$$

Proof. Since $0 < t_0 < t_{k+1} < \theta_k < t_k < \frac{1}{2}$, $k = 1, 2, \ldots$, then for any $k \in N$ and $u \in K$, by Lemma 3.2 we have

$$u^{(n-2)}(t) \geq \theta_k \|u\|, \quad t \in [\theta_k, 1-\theta_k]. \tag{4.1}$$

Consider the sequences $\{\Omega_{1,k}\}_{k=1}^{\infty}$ and $\{\Omega_{2,k}\}_{k=1}^{\infty}$ of open subsets of E defined by

$$\Omega_{1,k} = \{u \in K : \|u\| < r_k\}, \quad k = 1, 2, \ldots,$$
$$\Omega_{2,k} = \{u \in K : \|u\| < R_k\}, \quad k = 1, 2, \ldots .$$

For a fixed k and $u \in \partial\Omega_{1,k}$, from (4.1) we have

$$r_k = \|u\| \geq u^{(n-2)}(t) \geq \theta_k \|u\| = \theta_k r_k, \quad t \in [\theta_k, 1-\theta_k].$$

Since $(t_1, 1-t_1) \subset [\theta_k, 1-\theta_k]$, we consider the following three cases:

(i) If $\xi_1 \in [t_1, 1-t_1]$, then from (3.1), condition (C_5) and Lemma 3.1, we have

$$\begin{aligned}
\|Tu\| &= (Tu)^{(n-2)}(1) \\
&= \int_0^1 \frac{1}{p(s)}\varphi^{-1}\left(\int_s^1 a(\tau)f(u(\tau), u'(\tau), \ldots, u^{(n-2)}(\tau))\mathrm{d}\tau\right)\mathrm{d}s \\
&\quad + \frac{\sum_{i=1}^{m-2}\alpha_i \int_0^{\xi_i} \frac{1}{p(s)}\varphi^{-1}\left(\int_s^1 a(\tau)f(u(\tau), u'(\tau), \ldots, u^{(n-2)}(\tau))\mathrm{d}\tau\right)\mathrm{d}s}{1-\sum_{i=1}^{m-2}\alpha_i} \\
&\geq \int_{\xi_1}^{1-t_1} \frac{1}{p(s)}\varphi^{-1}\left(\int_s^{1-t_1} a(\tau)f(u(\tau), u'(\tau), \ldots, u^{(n-2)}(\tau))\mathrm{d}\tau\right)\mathrm{d}s \\
&\quad + \frac{\sum_{i=1}^{m-2}\alpha_i \int_{t_1}^{\xi_1} \frac{1}{p(s)}\varphi^{-1}\left(\int_s^{\xi_1} a(\tau)f(u(\tau), u'(\tau), \ldots, u^{(n-2)}(\tau))\mathrm{d}\tau\right)\mathrm{d}s}{1-\sum_{i=1}^{m-2}\alpha_i} \\
&\geq (mr_k)\left[\int_{\xi_1}^{1-t_1} \frac{1}{p(s)}\varphi^{-1}\left(\int_s^{1-t_1} a(\tau)\mathrm{d}\tau\right)\mathrm{d}s + \frac{\sum_{i=1}^{m-2}\alpha_i}{1-\sum_{i=1}^{m-2}\alpha_i}\int_{t_1}^{\xi_1} \frac{1}{p(s)}\varphi^{-1}\left(\int_s^{\xi_1} a(\tau)\mathrm{d}\tau\right)\mathrm{d}s\right] \\
&= mr_k H(\xi_1) > mr_k L > r_k = \|u\|.
\end{aligned}$$

(ii) If $\xi_1 \in [0, t_1]$, then from (3.1), condition (C_5) and Lemma 3.1, we have

$$\begin{aligned}
\|Tu\| &= (Tu)^{(n-2)}(1) \\
&= \int_0^1 \frac{1}{p(s)}\varphi^{-1}\left(\int_s^1 a(\tau)f(u(\tau), u'(\tau), \ldots, u^{(n-2)}(\tau))\mathrm{d}\tau\right)\mathrm{d}s \\
&\quad + \frac{\sum_{i=1}^{m-2}\alpha_i \int_0^{\xi_i} \frac{1}{p(s)}\varphi^{-1}\left(\int_s^1 a(\tau)f(u(\tau), u'(\tau), \ldots, u^{(n-2)}(\tau))\mathrm{d}\tau\right)\mathrm{d}s}{1-\sum_{i=1}^{m-2}\alpha_i} \\
&\geq \int_{t_1}^{1-t_1} \frac{1}{p(s)}\varphi^{-1}\left(\int_s^{1-t_1} a(\tau)f(u(\tau), u'(\tau), \ldots, u^{(n-2)}(\tau))\mathrm{d}\tau\right)\mathrm{d}s
\end{aligned}$$

$$\begin{aligned}
&\geq (mr_k)\left[\int_{t_1}^{1-t_1}\frac{1}{p(s)}\varphi^{-1}\left(\int_s^{1-t_1}a(\tau)\mathrm{d}\tau\right)\mathrm{d}s\right]\\
&= mr_kH(t_1) > mr_kL > r_k = \|u\|.
\end{aligned}$$

(iii) If $\xi_1 \in [1-t_1, 1]$, then from (3.1), condition (C_5) and Lemma 3.1, we have

$$\begin{aligned}
\|Tu\| &= (Tu)^{(n-2)}(1)\\
&= \int_0^1 \frac{1}{p(s)}\varphi^{-1}\left(\int_s^1 a(\tau)f(u(\tau),u'(\tau),\ldots,u^{(n-2)}(\tau))\mathrm{d}\tau\right)\mathrm{d}s\\
&\quad + \frac{\sum_{i=1}^{m-2}\alpha_i\int_0^{\xi_i}\frac{1}{p(s)}\varphi^{-1}\left(\int_s^1 a(\tau)f(u(\tau),u'(\tau),\ldots,u^{(n-2)}(\tau))\mathrm{d}\tau\right)\mathrm{d}s}{1-\sum_{i=1}^{m-2}\alpha_i}\\
&\geq \frac{\sum_{i=1}^{m-2}\alpha_i\int_{t_1}^{1-t_1}\frac{1}{p(s)}\varphi^{-1}\left(\int_s^{1-t_1} a(\tau)f(u(\tau),u'(\tau),\ldots,u^{(n-2)}(\tau))\mathrm{d}\tau\right)\mathrm{d}s}{1-\sum_{i=1}^{m-2}\alpha_i}\\
&\geq (mr_k)\left[\frac{\sum_{i=1}^{m-2}\alpha_i}{1-\sum_{i=1}^{m-2}\alpha_i}\int_{t_1}^{1-t_1}\frac{1}{p(s)}\varphi^{-1}\left(\int_s^{1-t_1}a(\tau)\mathrm{d}\tau\right)\mathrm{d}s\right]\\
&= mr_kH(1-t_1) > mr_kL > r_k = \|u\|.
\end{aligned}$$

Thus in all cases, Theorem 2.1 implies that

$$i(T,\ \Omega_{1,k},\ K) = 0. \tag{4.2}$$

On the other hand, let $u(t) \in \partial\Omega_{2,k}$, we have $u^{(n-2)}(t) \leq \|u\| = R_k$, and by ($C_6$) we have

$$\begin{aligned}
\|Tu\| &= (Tu)^{(n-2)}(1)\\
&= \int_0^1 \frac{1}{p(s)}\varphi^{-1}\left(\int_s^1 a(\tau)f(u(\tau),u'(\tau),\ldots,u^{(n-2)}(\tau))\mathrm{d}\tau\right)\mathrm{d}s\\
&\quad + \frac{\sum_{i=1}^{m-2}\alpha_i\int_0^{\xi_i}\frac{1}{p(s)}\varphi^{-1}\left(\int_s^1 a(\tau)f(u(\tau),u'(\tau),\ldots,u^{(n-2)}(\tau))\mathrm{d}\tau\right)\mathrm{d}s}{1-\sum_{i=1}^{m-2}\alpha_i}\\
&\leq \int_0^1 \frac{1}{p(0)}\varphi^{-1}\left(\int_s^1 a(\tau)f(u(\tau),u'(\tau),\ldots,u^{(n-2)}(\tau))\mathrm{d}\tau\right)\mathrm{d}s\\
&\quad + \frac{\sum_{i=1}^{m-2}\alpha_i\int_0^{\xi_{m-2}}\frac{1}{p(0)}\varphi^{-1}\left(\int_s^1 a(\tau)f(u(\tau),u'(\tau),\ldots,u^{(n-2)}(\tau))\mathrm{d}\tau\right)\mathrm{d}s}{1-\sum_{i=1}^{m-2}\alpha_i}\\
&\leq \frac{MR_k}{p(0)}\frac{1}{1-\sum_{i=1}^{m-2}\alpha_i}\left[\int_0^1\varphi^{-1}\left(\int_s^1 a(\tau)\mathrm{d}\tau\right)\mathrm{d}s\right]\\
&= R_k = \|u\|.
\end{aligned}$$

Thus Theorem 2.1 implies that

$$i(T,\ \Omega_{2,k},\ K) = 1. \tag{4.3}$$

Hence, since $r_k < R_k$ for $k \in N$, (4.2) and (4.3), it follows from the additivity of the fixed point index that

$$i(T,\ \Omega_{2,k}\setminus\overline{\Omega}_{1,k},\ K) = 1, \quad \text{for } k \in N.$$

Thus T has a fixed point in $\Omega_{2,k}\setminus\overline{\Omega}_{1,k}$ such that $r_k \leq \|u_k\| \leq R_k$. Since $k \in N$ was arbitrary, the proof is complete. $\square$

In order to use Theorem 2.2, we let $\theta_k < r_k < 1-\theta_k$ with θ_k of Theorem 4.1, and define the nonnegative, increasing, continuous functionals γ_k, β_k, and α_k by

$$\gamma_k(u) = \max_{\theta_k \le t \le r_k} u^{(n-2)}(t) = u^{(n-2)}(r_k),$$
$$\beta_k(u) = \min_{r_k \le t \le 1-\theta_k} u^{(n-2)}(t) = u^{(n-2)}(r_k),$$
$$\alpha_k(u) = \max_{\theta_k \le t \le 1-\theta_k} u^{(n-2)}(t) = u^{(n-2)}(1-\theta_k).$$

It is obvious that for each $u \in K$,

$$\gamma_k(u) \le \beta_k(u) \le \alpha_k(u).$$

In addition, by Lemma 3.3, for each $u \in K$,

$$\gamma_k(u) = u^{(n-2)}(r_k) \ge r_k\|u\|.$$

Thus

$$\|u\| \le \frac{1}{r_k}\gamma_k(u), \quad \text{for all } u \in K.$$

For convenience, we denote

$$\lambda = \frac{1}{p(0)} \frac{1}{1-\sum_{i=1}^{m-2}\alpha_i} \left[\int_0^1 \varphi^{-1}\left(\int_s^1 a(\tau)\mathrm{d}\tau\right)\mathrm{d}s\right],$$
$$\eta_k = \frac{\theta_k}{p(1)}\left[\varphi^{-1}\left(\int_{1-\theta_k}^1 a(\tau)\mathrm{d}\tau\right)\right].$$

Theorem 4.2. *Suppose that conditions* (C_1)–(C_4) *hold. Let* $\{\theta_k\}_{k=1}^{\infty}$ *be such that* $\theta_k \in (t_{k+1}, t_k)$ $(k = 1, 2, \ldots)$. *Let* $\{a_k\}_{k=1}^{\infty}$, $\{b_k\}_{k=1}^{\infty}$ *and* $\{c_k\}_{k=1}^{\infty}$ *be such that*

$$c_{k+1} < a_k < r_k b_k < \frac{r_k\lambda}{\eta_k} b_k < c_k.$$

Furthermore, for each natural number k assume that f satisfies:

(C_7) $f(v_1, v_2, \ldots, v_{n-1}) < \varphi\left(\frac{c_k}{\lambda}\right)$, *for all* $0 \le v_1, v_2, \ldots, v_{n-1} \le \frac{c_k}{r_k}$,

(C_8) $f(v_1, v_2, \ldots, v_{n-1}) > \varphi\left(\frac{b_k}{\eta_k}\right)$, *for all* $0 \le v_1, v_2, \ldots, v_{n-2} \le \frac{b_k}{r_k}$, $b_k \le v_{n-1}(t) \le \frac{b_k}{r_k}$,

(C_9) $f(v_1, v_2, \ldots, v_{n-1}) < \varphi\left(\frac{a_k}{\lambda}\right)$, *for all* $0 \le v_1, v_2, \ldots, v_{n-1} \le \frac{a_k}{r_k}$.

Then the boundary value problem (1.1) and (1.2) has three infinite families of solutions $\{u_{1k}\}_{k=1}^{\infty}$, $\{u_{2k}\}_{k=1}^{\infty}$ *and* $\{u_{3k}\}_{k=1}^{\infty}$ *satisfying*

$$0 \le \alpha_k(u_{1k}) < a_k < \alpha_k(u_{2k}), \qquad \beta_k(u_{2k}) < b_k < \beta_k(u_{3k}), \qquad \gamma(u_{3k}) < c_k, \quad \text{for } n \in N.$$

Proof. We define the completely continuous operator T by (3.1). So it is easy to check that $T : \overline{K(\gamma_k, c_k)} \to K$, for $k \in N$.

We now show that all the conditions of Theorem 2.2 are satisfied. To make use of property (i) of Theorem 2.2, we choose $u \in \partial K(\gamma_k, c_k)$. Then

$$\gamma_k(u) = \max_{\theta_k \le t \le r_k} u^{(n-2)}(t) = u^{(n-2)}(r_k) = c_k.$$

This implies that $0 \le u^{(n-2)} \le c_k$ for $[0, r_k]$. If we recall that $\|u\| \le \frac{1}{r_k}\gamma_k(u) = \frac{1}{r_k}c_k$. Then we have

$$0 \le u^{(i)}(t) \le \frac{c_k}{r_k}, \quad 0 \le t \le 1,\ i = 0, 1, \ldots, n-1.$$

Thus assumption (C_7) implies

$$f(u(t), u'(t), \ldots, u^{(n-2)}(t)) < \varphi\left(\frac{c_k}{\lambda}\right), \quad 0 \le t \le 1.$$

Therefore

$$\begin{aligned}
\gamma_k(Tu) &= \max_{\theta_k \le t \le r_k} (Tu)^{(n-2)}(t) = (Tu)^{(n-2)}(r_k) \\
&= \int_0^{r_k} \frac{1}{p(s)} \varphi^{-1}\left(\int_s^1 a(\tau) f(u(\tau), u'(\tau), \ldots, u^{(n-2)}(\tau)) \mathrm{d}\tau\right) \mathrm{d}s \\
&\quad + \frac{\sum_{i=1}^{m-2} \alpha_i \int_0^{\xi_i} \frac{1}{p(s)} \varphi^{-1}\left(\int_s^1 a(\tau) f(u(\tau), u'(\tau), \ldots, u^{(n-2)}(\tau)) \mathrm{d}\tau\right) \mathrm{d}s}{1 - \sum_{i=1}^{m-2} \alpha_i} \\
&\le \int_0^{1} \frac{1}{p(0)} \varphi^{-1}\left(\int_s^1 a(\tau) f(u(\tau), u'(\tau), \ldots, u^{(n-2)}(\tau)) \mathrm{d}\tau\right) \mathrm{d}s \\
&\quad + \frac{\sum_{i=1}^{m-2} \alpha_i \int_0^{\xi_{m-2}} \frac{1}{p(0)} \varphi^{-1}\left(\int_s^1 a(\tau) f(u(\tau), u'(\tau), \ldots, u^{(n-2)}(\tau)) \mathrm{d}\tau\right) \mathrm{d}s}{1 - \sum_{i=1}^{m-2} \alpha_i} \\
&\le \frac{c_k}{\lambda p(0)} \frac{1}{1 - \sum_{i=1}^{m-2} \alpha_i} \left[\int_0^1 \varphi^{-1}\left(\int_s^1 a(\tau) \mathrm{d}\tau\right) \mathrm{d}s\right] \\
&= c_k.
\end{aligned}$$

Hence condition (i) is satisfied.

Secondly, we show that (ii) of Theorem 2.2 is fulfilled. For this we select $u \in \partial K(\beta_k, b_k)$. Then $\beta_k(u) = \min_{r_k \le t \le 1-\theta_k} u^{(n-2)}(t) = u^{(n-2)}(r_k) = b_k$. This fact implies that $u^{(n-2)}(t) \ge b_k$, for $r_k \le t \le 1$. Noticing that $\|u\| \le \frac{1}{r_k}\gamma_k(u) \le \frac{1}{r_k}\beta_k(u) = \frac{b_k}{r_k}$, we have

$$b_k \le u^{(n-2)}(t) \le \frac{b_k}{r_k}, \quad \text{for } r_k \le t \le 1.$$

By (C_8), we have

$$f(u(t), u'(t), \ldots, u^{(n-2)}(t)) > \varphi\left(\frac{b_k}{\eta_k}\right).$$

Therefore

$$\begin{aligned}
\beta_k(Tu) &= \min_{r_k \le t \le 1-\theta_k} (Tu)^{(n-2)}(t) = (Tu)^{(n-2)}(r_k) \\
&= \int_0^{r_k} \frac{1}{p(s)} \varphi^{-1}\left(\int_s^1 a(\tau) f(u(\tau), u'(\tau), \ldots, u^{(n-2)}(\tau)) \mathrm{d}\tau\right) \mathrm{d}s \\
&\quad + \frac{\sum_{i=1}^{m-2} \alpha_i \int_0^{\xi_i} \frac{1}{p(s)} \varphi^{-1}\left(\int_s^1 a(\tau) f(u(\tau), u'(\tau), \ldots, u^{(n-2)}(\tau)) \mathrm{d}\tau\right) \mathrm{d}s}{1 - \sum_{i=1}^{m-2} \alpha_i} \\
&\ge \int_0^{r_k} \frac{1}{p(s)} \varphi^{-1}\left(\int_{r_k}^1 a(\tau) f(u(\tau), u'(\tau), \ldots, u^{(n-2)}(\tau)) \mathrm{d}\tau\right) \mathrm{d}s \\
&\ge \frac{b_k}{\eta_k} \frac{\theta_k}{p(1)} \left[\varphi^{-1}\left(\int_{1-\theta_k}^1 a(\tau) \mathrm{d}\tau\right)\right] \\
&= b_k.
\end{aligned}$$

Hence condition (ii) is satisfied.

Finally, we verify that (iii) of Theorem 2.2 is also satisfied. Noting that $u^{(n-2)}(t) \equiv \frac{a_k}{4}, 0 \le t \le 1$ is a member of $K(\alpha_k, a_k)$ and $\alpha_k(u) = \frac{a_k}{4} < a_k$. We have $K(\alpha_k, a_k) \ne \emptyset$. Now let $u \in \partial K(\alpha_k, a_k)$. Then $\alpha_k(u) = \max_{\theta_k \le t \le 1-\theta_k} u^{(n-2)}(t) = u^{(n-2)}(1-\theta_k) = a_k$. This implies that $0 \le u^{(n-2)}(t) \le a_k, 0 \le t \le 1-\theta_k$. Noticing that $\|u\| \le \frac{1}{r_k}\gamma_k(u) \le \frac{1}{r_k}\alpha_k(u) = \frac{a_k}{r_k}$, we get

$$0 \le u^{(i)}(t) \le \frac{a_k}{r_k}, \quad 0 \le t \le 1,\ i = 0, 1, \dots, n-1.$$

Then assumption (C_9) implies

$$f(u(t), u'(t), \dots, u^{(n-2)}(t)) < \varphi\left(\frac{a_k}{\lambda}\right), \quad 0 \le t \le 1.$$

As before, we get

$$\begin{aligned}
\alpha_k(Tu) &= \max_{\theta_k \le t \le 1-\theta_k} (Tu)(t) = (Tu)(1-\theta_k) \\
&= \int_0^{1-\theta_k} \frac{1}{p(s)} \varphi^{-1}\left(\int_s^1 a(\tau) f(u(\tau), u'(\tau), \dots, u^{(n-2)}(\tau)) d\tau\right) ds \\
&\quad + \frac{\sum_{i=1}^{m-2} \alpha_i \int_0^{\xi_i} \frac{1}{p(s)} \varphi^{-1}\left(\int_s^1 a(\tau) f(u(\tau), u'(\tau), \dots, u^{(n-2)}(\tau)) d\tau\right) ds}{1 - \sum_{i=1}^{m-2} \alpha_i} \\
&\le \int_0^1 \frac{1}{p(0)} \varphi^{-1}\left(\int_s^1 a(\tau) f(u(\tau), u'(\tau), \dots, u^{(n-2)}(\tau)) d\tau\right) ds \\
&\quad + \frac{\sum_{i=1}^{m-2} \alpha_i \int_0^{\xi_{m-2}} \frac{1}{p(0)} \varphi^{-1}\left(\int_s^1 a(\tau) f(u(\tau), u'(\tau), \dots, u^{(n-2)}(\tau)) d\tau\right) ds}{1 - \sum_{i=1}^{m-2} \alpha_i} \\
&\le \frac{a_k}{\lambda p(0)} \frac{1}{1 - \sum_{i=1}^{m-2} \alpha_i} \left[\int_0^1 \varphi^{-1}\left(\int_s^1 a(\tau) d\tau\right) ds\right] \\
&= a_k.
\end{aligned}$$

Thus (iii) of Theorem 2.2 is satisfied. Since all hypotheses of Theorem 2.2 are satisfied, the assertion follows. □

Remark 4.1. If we add the condition $a(t)f(u(t), u'(t), \dots, u^{(n-2)}(t)) \not\equiv 0, t \in [0, 1]$, to Theorem 4.2, we can get three infinite families of positive solutions $\{u_{1k}\}_{k=1}^{\infty}$ $\{u_{2k}\}_{k=1}^{\infty}$ and $\{u_{3k}\}_{k=1}^{\infty}$ satisfying

$$0 < \alpha_k(u_{1k}) < a_k < \alpha_k(u_{2k}), \qquad \beta_k(u_{2k}) < b_k < \beta_k(u_{3k}), \qquad \gamma(u_{3k}) < c_k, \quad \text{for } n \in N.$$

5. Examples and remark

There exists a function $a(t)$ satisfying condition (C_4).

Example 5.1. Let

$$\delta = \left(\frac{\pi^2}{3} - \frac{9}{4}\right), \qquad t^* = \frac{15}{32}, \qquad t_i = t^* - \sum_{k=1}^{i} \frac{1}{2(k+1)^4}, \quad i = 1, 2, \dots.$$

Consider the function $a(t) : [0, 1] \to (0, \infty)$, $a(t) = \sum_{i=1}^{\infty} a_i(t),\ t \in [0, 1]$, where

$$a_i(t) = \begin{cases} \dfrac{2}{(2i-1)(2i+1)(t_{i+1}+t_i)}, & 0 \le t < \dfrac{t_{i+1}+t_i}{2}, \\ \dfrac{1}{\delta(t_i - t)^{\frac{1}{2}}}, & \dfrac{t_{i+1}+t_i}{2} \le t < t_i, \\ \dfrac{1}{\delta(t - t_i)^{\frac{1}{2}}}, & t_i < t \le \dfrac{t_i + t_{i-1}}{2}, \\ \dfrac{2}{(2i-1)(2i+1)(2 - t_i - t_{i-1})}, & \dfrac{t_i + t_{i-1}}{2} < t \le 1. \end{cases}$$

It is easy to see that $t_1 = \frac{7}{16} < \frac{1}{2},\ t_i - t_{i+1} = \frac{1}{2(i+2)^4},\ i = 1, 2, \ldots$ (note $\sum_{i=1}^{\infty} \frac{1}{i^4} = \frac{\pi^4}{90}$), and

$$t_0 = \lim_{i\to\infty} t_i = \frac{15}{32} - \sum_{k=1}^{\infty} \frac{1}{2(k+1)^4} = \frac{31}{32} - \frac{\pi^4}{180} > \frac{1}{5}$$

and because $\sum_{i=1}^{\infty} \frac{1}{i^2} = \frac{\pi^2}{6}$, we have

$$\begin{aligned}\sum_{i=1}^{\infty} \int_0^1 a_i(t)\mathrm{d}t &= \sum_{i=1}^{\infty} \frac{2}{(2i-1)(2i+1)} + \frac{1}{\delta} \sum_{i=1}^{\infty} \left[\int_{\frac{t_{i+1}+t_i}{2}}^{t_i} \frac{1}{(t_i - t)^{\frac{1}{2}}}\mathrm{d}t + \int_{t_i}^{\frac{t_i+t_{i-1}}{2}} \frac{1}{(t-t_i)^{\frac{1}{2}}}\mathrm{d}t \right] \\ &= 1 + \frac{\sqrt{2}}{\delta} \sum_{i=1}^{\infty} \left[(t_i - t_{i+1})^{\frac{1}{2}} + (t_{i-1} - t_i)^{\frac{1}{2}} \right] \\ &= 1 + \frac{1}{\delta} \sum_{i=1}^{\infty} \left[\frac{1}{(i+2)^2} + \frac{1}{(i+1)^2} \right] \\ &= 1 + \frac{1}{\delta} \left[\frac{\pi^2}{3} - \frac{9}{4} \right] = 2.\end{aligned}$$

Therefore

$$\int_0^1 a(t)\mathrm{d}t = \sum_{i=1}^{\infty} \int_0^1 a_i(t)\mathrm{d}t = 2 < \infty.$$

This implies Condition (C_4).

Example 5.2. As an example we mention the boundary value problem

$$\begin{cases} (\varphi(p(t)u^{(n-1)}(t))(t))' + a(t)f(u(t), u'(t), \ldots, u^{(n-2)}(t)) = 0, & 0 < t < 1, \\ u^{(i)}(0) = 0, \quad i = 0, 1, \ldots, n-3, \\ u^{(n-2)}(0) = \sum_{i=1}^{m-2} \alpha_i u^{(n-2)}(\xi_i), \qquad u^{(n-1)}(1) = 0, \end{cases} \tag{5.1}$$

where

$$\varphi(u) = \begin{cases} \dfrac{u^3}{1+u^5}, & u \le 0, \\ u^6 & u > 0. \end{cases}$$

Here $\alpha_i,\ a(t) \in ((0,\ 1), [0, \infty))$ and f satisfies the conditions of Theorem 4.1. It is clear that $\varphi : R \to R$ is an increasing homeomorphism and positive homomorphism and $\varphi(0) = 0$.

Remark 5.1. Because p-Laplacian operators are odd, they do not apply to our example. Hence we generalize boundary value problems p-Laplacian operators and the results [3,6,7,10] do not apply to the example.

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